

# Wronskian perturbation theory

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**Abstract.** We develop a perturbation method that generalizes an approach proposed recently to treat velocity-dependent quantum-mechanical models. In order to test present approach we apply it to some simple trivial and nontrivial examples.

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## 1 Introduction

In a recent paper Jaghoub [1] developed a perturbation theory for velocity-dependent quantum-mechanical models. Such potentials are useful, for example, for the study of pion-nucleon scattering and in models of particles with coordinate-dependent masses [1] (and references therein). According to the author, one of the advantages of the method is that its main equations depend only on the eigenfunction of the chosen unperturbed state. More precisely, the method does not need neither the whole unperturbed energy spectrum nor the basis set of unperturbed eigenfunctions as in the formulation of the Rayleigh-Schrödinger perturbation theory in terms of sums over intermediate

states. Jaghoub chose rather too simple examples in order to test his method [1].

Similar perturbation approaches are known since long ago, but they apply mainly to local (coordinate-dependent) potentials [2]. Although one can in principle adapt some of the well-known perturbation methods [3] to the treatment of velocity-dependent potentials, here we proceed in a different way.

The purpose of this paper is to generalize the method proposed by Jaghoub [1] and derive a perturbation algorithm for the treatment of a wider variety of problems. In Sec. 2 we develop our version of the method in a quite general way. In Sec. 3 we apply the perturbation approach to an exactly solvable example which enables us to compare

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the results of present procedure with the expansion of the exact eigenfunctions and eigenvalues. In Sec. 4 we discuss a partially solvable example treated by Jaghoub [1]. Finally, in Sec. 5 we summarize our main results and discuss other potential applications of present method.

## 2 Method

Suppose that we want to solve the differential equation

$$y''(x) = F(y, x) \quad (1)$$

where the prime denotes differentiation with respect to  $x$  and  $F(y, x)$  an arbitrary linear differential operation on  $y(x)$ . In order to apply perturbation theory we choose a closely related, solvable problem

$$y_0''(x) = F_0(y_0, x). \quad (2)$$

If we multiply Eq. (1) by  $y_0$  and subtract Eq. (2) multiplied by  $y$  we obtain

$$\frac{d}{dx} W(y, y_0) = F y_0 - F_0 y \quad (3)$$

where  $W(y, y_0)$  stands for the Wronskian

$$W(y, y_0) = y' y_0 - y'_0 y = y_0^2 \left( \frac{y}{y_0} \right)'. \quad (4)$$

On integrating Eq. (3) twice we obtain

$$y(x) = C_2 y_0(x) + C_1 y_0(x) \int_{\beta}^x \frac{dx'}{y_0(x')^2} + y_0(x) \int_{\beta}^x \frac{dx'}{y_0(x')^2} \int_{\alpha}^{x'} (F y_0 - F_0 y) (x'') dx'' \quad (5)$$

where the integration constants  $C_1$  and  $C_2$  and the integration limits  $\alpha$  and  $\beta$  enable one to accommodate the boundary conditions of the problem and the normalization of the solution.

Notice that the function

$$u(x) = y_0(x) \int_{\beta}^x \frac{dx'}{y_0(x')^2} \quad (6)$$

satisfies  $W(u, y_0) = 1$ , so that  $u(x)$  does not vanish and is finite at the zeroes of  $y_0(x)$ . If the operation  $F_0$  is simply of the form  $F_0(y_0, x) = f_0(x)y_0(x)$ , then  $y_0(x)$  and  $u(x)$  are two linearly independent solutions of Eq. (2), and  $u(x)$  is called ghost state [4] (and references therein). A complex linear combination of  $y_0(x)$  and  $u(x)$  proved suitable for the construction of a logarithmic perturbation method for excited states [4].

In order to apply perturbation theory we introduce a perturbation parameter  $\lambda$  into  $F$  and expand

$$F = \sum_{j=0}^{\infty} F_j \lambda^j \quad (7)$$

and

$$y(x) = \sum_{j=0}^{\infty} y_j(x) \lambda^j. \quad (8)$$

We simply introduce the series (7) and (8) into Eq. (5) and obtain  $y_j(x)$  in terms of  $y_k(x)$ ,  $k = 0, 1, \dots, j-1$ , and  $E_k$ ,  $k = 1, 2, \dots, j$ . The boundary conditions determine the appropriate value of  $E_j$  that is the only unknown in the expression of  $y_j(x)$ . The procedure will be made more explicit in the examples below.

Present approach applies to bound and unbound states; in this paper we concentrate on the former ones. It is customary to choose a convenient normalization for bound states. Here we keep our equations as simple as possible and add a normalization factor  $N(\lambda) = N_0 + N_1 \lambda + \dots$  at the end of the calculation if necessary. We may, for example, require that  $(N_0 + N_1 \lambda + \dots)(y_0 + y_1 \lambda + \dots)$  be

normalized to unity up to a given perturbation order. Assuming that  $y_0(x)$  is normalized to unity we easily prove that

$$N(\lambda) = 1 - \lambda \int y_0 y_1 dx + \frac{\lambda^2}{2} \left[ 3 \left( \int y_0 y_1 dx \right)^2 \right. \quad (9)$$

$$\left. - 2 \int y_0 y_2 dx - \int y_1^2 dx \right] + \dots, \quad (10)$$

where the integrals extend over the whole physical coordinate range. Obviously, the eigenvalues are independent of the normalization constant.

It is worth noticing that we have not made explicit use of the assumed linearity of  $F(y, x)$  in order to derive Eq. (5). Consequently, that equation applies to arbitrary nonlinear differential operators  $F$  and  $F_0$ . However, the eigenvalues of nonlinear equations do depend on the chosen normalization condition and we have to take this fact explicitly into account in order to solve them.

### 3 Solvable example

In order to test the general equations developed above we choose the eigenvalue problem

$$\begin{aligned} y''(x) &= \lambda y'(x) - E y(x), \\ y(0) &= y(1) = 0 \end{aligned} \quad (11)$$

with solutions

$$\begin{aligned} y(x) &= A e^{\lambda x/2} \sin(n\pi x) \\ E &= n^2 \pi^2 + \frac{\lambda^2}{4}, \quad n = 1, 2, \dots \end{aligned} \quad (12)$$

Notice that the eigenvalue equation (11) is not Hermitian but it supports real eigenvalues for real  $\lambda$ . For the time

being the exact form of the normalization factor  $A$  is unnecessary for present perturbation calculation and we can add it easily at the end of the process as discussed above.

In order to simplify the presentation and discussion of the results we arbitrarily choose the normalization factor for the unperturbed solution:  $A = A(\lambda = 0) = \sqrt{2}$ .

Straightforward expansion of the exact results (12) in a Taylor series about  $\lambda = 0$  yields

$$\begin{aligned} y_j(x) &= \frac{x^j}{j! 2^j} y_0(x) \\ E_j(n) &= n^2 \pi^2 \delta_{j0} + \frac{1}{4} \delta_{j2}. \end{aligned} \quad (13)$$

In order to apply present perturbation theory to this simple test model we make the obvious choice  $F = \lambda y'(x) - E y(x)$  and  $F_0 = -E_0 y_0(x)$ , where  $E_0(n) = n^2 \pi^2$  and  $y_0(n, x) = \sqrt{2} \sin(n\pi x)$ . Since  $u(x)$  does not vanish at the end points we choose  $C_1 = 0$ . The other integration constant  $C_2$  is important to force a normalization condition at each order of the perturbation algorithm. However, as in this first step we are not interested in the normalization of the solution we simply set  $C_2 = 0$ . Thus we are left with the hierarchical perturbation equations

$$y_j(x) = y_0(x) \int_0^x \frac{dx'}{y_0(x')^2} \int_0^{x'} (y_0 y'_{j-1} \quad (14)$$

$$- y_0 \sum_{k=1}^j E_k y_{j-k}) (x'') dx'', \quad (15)$$

where we have chosen  $\alpha = \beta = 0$  to satisfy the boundary condition at  $x = 0$ . Notice that  $y_j$  depends on  $E_j$  that is determined by the boundary condition at  $x = 1$ . For example, at first order we obtain

$$y_1(x) = \frac{\sqrt{2} E_1 x \cos(n\pi x)}{2n\pi} + \left( \frac{\sqrt{2} x}{2} - \frac{\sqrt{2} E_1}{2n^2 \pi^2} \right) \sin(n\pi x) \quad (16)$$

that does not satisfy the boundary condition at  $x = 1$  unless  $E_1 = 0$ . Proceeding exactly in the same way we obtain the function and energy perturbation coefficients (13) order by order, which clearly shows that Eq. (15) gives the correct answer.

The normalization factor calculated by perturbation theory

$$N(\lambda) = 1 - \frac{\lambda}{4} + \frac{\lambda^2(n^2\pi^2 + 12)}{96n^2\pi^2} + \frac{\lambda^3(n^2\pi^2 - 12)}{384n^2\pi^2} + \dots \quad (17)$$

also agrees with the one derived from the exact solution.

## 4 Partially solvable example

In order to compare our approach with Jaghoub's one [1] more closely, in what follows we treat one of that author's examples:

$$\left(1 - \frac{3x^2}{5}\right)y''(x) - \frac{6}{5}[xy'(x) - y(x)] + Ey(x) = 0, \quad (18)$$

where  $y(0) = y(1) = 0$  exactly as in the preceding example. Jaghoub constructed this problem in order to have an exact solution for the ground state [1]

$$\begin{aligned} y(x) &= Ax(1 - x^2), \\ E(n = 1) &= 6 \end{aligned} \quad (19)$$

and treated only this trivial case by perturbation theory. Here, on the other hand, we also consider the excited states for which there are no exact solutions as far as we know. A straightforward calculation with  $y_0(x) = \sin(n\pi x)$ ,  $F = 3x^2y''/5 + 6(xy' - y)/5 - Ey$ , and  $F_0 = -E_0y_0$  shows that

$$E_1 = -\frac{2n^2\pi^2 + 15}{10},$$

$$\begin{aligned} y_1(x) &= \frac{n\pi x(x^2 - 1)\cos(n\pi x)}{10} + \left(\frac{3x^2}{20} + \frac{1}{10}\right)\sin(n\pi x), \\ E_2 &= -\frac{3(8n^4\pi^4 + 10n^2\pi^2 - 15)}{1000n^2\pi^2}, \\ E_3 &= -\frac{248n^6\pi^6 + 462n^4\pi^4 - 1575n^2\pi^2 + 1890}{35000n^4\pi^4}. \end{aligned} \quad (20)$$

The energy coefficients agree with those shown numerically by Jaghoub [1] for  $n = 1$ . In this case the normalization factor is given by

$$N(\lambda) = \sqrt{2} \left[ 1 - \frac{3\lambda}{20} - \frac{3(29n^4\pi^4 + 10n^2\pi^2 + 30)\lambda^2}{4000n^4\pi^4} + \dots \right] \quad (21)$$

We do not show the analytical expressions of  $y_2(x)$  and  $y_3(x)$ , as well as perturbation corrections of larger order, because they are rather complicated.

## 5 Conclusions

In this paper we have shown how to develop a useful perturbation method from the Wronskian between the perturbed and unperturbed solutions. The approach enables one to treat velocity-dependent quantum-mechanical problems as well as local perturbation potentials. Present approximation is more general than one proposed earlier [1] and enables us to discuss mathematical aspects of the solutions that could be otherwise masked. We easily derive the most general equation for the application of perturbation theory and can clearly analyze all the contributions to the approximate solutions and their behaviour regarding normalization and boundary conditions.

We have tested our general equations on an exactly solvable model showing that the approximate method yields exactly the same results that one obtains from expansion

of the exact solutions. We have also treated a partially solvable problem. In both cases we have derived perturbation corrections for all the states in terms of the quantum number, thus generalizing Jaghoub's results for the latter model [1].

Present perturbation method also applies to nonlinear models. We may treat them exactly as the examples above, except that one has to consider the normalization condition explicitly at every perturbation order because the eigenvalues depend on it.

For simplicity we have chosen the unperturbed and perturbed equations in such a way that we could solve all the integrals analytically. In more difficult cases one should have to resort to numerical integration. However, any problem that can be treated exactly by Jaghoub's approach can also be treated exactly by present one, probably in a more general way.

By means of an appropriate choice of the boundary conditions present perturbation approach is suitable for the approximate calculation of scattering phase shifts. In this field our method may be an alternative to the logarithmic perturbation theory that is commonly applied to local potentials [5,6].

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